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SELF-INTERSECTION OF THE COMPLEX SEPARATRICES AND THE NONEXISTENCE OF THE INTEGRALS IN THE HAMILTONIAN SYSTEMS WITH ONE-AND-HALF DEGREES OF FREEDOM*

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The relation connecting the branching of solutions (as functions of complex time) of the Hamiltonian system with two degrees of freedom and almost integrable, with the nonexistence of its supplementary first integral, were studied in /1,2/. In /2/ it was shown that both phenomena are related to the well known occurrence of the splitting of separatrices. Below, a related phenomenon is established connecting the branching and nonintegrability, namely the self-intersecting of the complex separatrices. The case of one-and-half degrees of freedom is considered for simplicity, as the case of two degrees of freedom can be treated in an analogous manner.

Let (M^2, ω^2) be a two-dimensional, connected complex analytic simplex manifold, $N^3 = M^2 \times S_c^1$ be a direct product of M^2 and the complex circumference S_c^1 with coordinate $t \,(\mathrm{mod}\, 2\pi)$ (i.e. the complex straight line C^1 with identified points differing from each other by a multiple of 2π) and $H(x, t, \mu) = H_0(x) + \mu H_1(x, t) + o(\mu) \, (x \in M)$ denotes a single-valued analytic function on N^3, analytically dependent on the parameter μ, $|\mu| < \mu_0$.

Consider in N^3 the Hamiltonian system

$$x^{\cdot} = IdH(x; t, \mu), \quad t^{\cdot} = 1, \quad (x, t) \in N^3 \tag{1}$$

where $dH(x; t, \mu)$ is the differential of H at fixed t and μ. Let $x_0 \in M$ be a fixed hyperbolic point of the system

$$x^{\cdot} = IdH_0(x), \quad x \in M \tag{2}$$

i.e. $d_{x_0} H_0 = 0$ and the eigenvalues $\pm\lambda$ of the system (2) linearized at the point x_0 have nonzero real parts, $\mathrm{Re}\,\lambda > 0$. Then, for sufficiently small $|\mu|$ the system (1) has a 2π-periodic solution $x = x_p(t, \mu)$ depending analytically on μ, $x_p(t, 0) = x_0$. Continuing the solutions of the system (1) asymptotic to $x = x_p(t, \mu)$ as $t \to \infty$ and maximally analytic in $t \in C$ (in general, non-uniquely), we obtain a two-dimensional complex surface $S_\mu \in N^3$ which shall be called the emergent complex separatrix of the periodic solution $x = x_p(t, \mu)$.

It turns out that the complex separatrix, unlike the real one, can have transversal self-intersections preventing the existence of an analytic (single-valued) first integral of the system (1).

Let us formulate the sufficient conditions for such a self-intersection. Let the asymptotic solution $x = x^*(t)$ of the system (2), $\lim x^*(t) = x_0$, $t \to -\infty$ be analytically continued along the (continuous) path $\Gamma: [0, 1] \to C$, $\Gamma(0) = \Gamma(1) \,(\mathrm{mod}\, 2\pi) \in R$ while $x^*(\Gamma(0)) = x^*(\Gamma(1))$. Then, for sufficiently small $|\mu|$ the solution $x = \varphi(t, t_0, \mu)$ of the system (1) with initial condition $\varphi(\Gamma(0) + t_0, t_0, \mu) = x^*(\Gamma(0))$ will be analytically continued along the path $(\Gamma + t_0): [0, 1] \to C$, $(\Gamma + t_0)(s) = \Gamma(s) + t_0$. Let $h(t_0, \mu) = H_0(\varphi(\Gamma(1) + t_0), (t_0, \mu)) - H_0(x^*(\Gamma(0))) = \mu h^1(t_0) + o(\mu)$ be the increment in the value of function $H_0(\varphi(t, t_0, \mu))$ when t varies along $\Gamma + t_0$.

Theorem. If the function h^1 has a simple zero, then for sufficiently small $|\mu| \neq 0$ the complex separatrix S_μ has a transversal self-intersection and the system (1) has no analytic first integral in N^3.

Note. The quantity $h^1(t_0)$ can be found from the formula

$$h^1(t_0) = \int_\Gamma \frac{\partial H_1}{\partial t}(x^*(t), t + t_0)\, dt \tag{3}$$

Indeed

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$$h(t_0,\mu)=\int_{\Gamma+t_0}\{H_0,H\}(\varphi(t,t_0,\mu),t,\mu)\,dt=\mu\int_{\Gamma+t_0}\{H_0,H_1\}(\varphi(t,t_0,0),t)\,dt+o(\mu)=$$
$$\mu\int_{\Gamma}\{H_0,H_1\}(x^*(t),t+t_0)\,dt+o(\mu)=\mu\int_{\Gamma}\left[\frac{\partial H_1}{\partial t}(x^*(t),t+t_0)-\frac{d}{dt}H_1(x^*(t),t+t_0)\right]dt+o(\mu)=$$
$$\mu\int_{\Gamma}\frac{\partial H_1}{\partial t}(x^*(t),t+t_0)\,dt+o(\mu)$$

from which follows (3). Here $\{\ ,\ \}$ denote the Poisson's brackets.

Proof of the theorem. Let $\Pi=\{t\in S_c^1\,/\,|\operatorname{Im} t|<A\}$, $(A>0)$ be a strip containing the simple zero of the function h^1. According to the Moser theorem /3/ an analytic simplex change of variables exists, depending analytically on μ

$$x=\Phi(\xi,\eta,t,\mu)=\Phi_0(\xi,\eta)+\mu\Phi_1(\xi,\eta,t)+o(\mu) \tag{4}$$
$$\Phi_0(0,0)=x_0,\ t\in\Pi,\ |\xi|<\varepsilon,\ |\eta|<\varepsilon,\ |\mu|<\varepsilon\ (\varepsilon<0)$$

and reducing the system (1) in the region $U_\mu=\{(x,t)\in N^3\mid t\in\Pi,\ x=\Phi(\xi,\eta,t,\mu),\ |\xi|<\varepsilon,\ |\eta|<\varepsilon\}$ to the normal form

$$\xi^\cdot=F_\eta,\ \eta^\cdot=-F_\xi,\ F(\xi,\eta,\mu)=G(\omega,\mu)=G_0(\omega)+O(\mu),\ \omega=\xi\eta,\ G_0'(0)=\lambda \tag{5}$$

The system (5) can be integrated

$$\omega=\omega_0,\ \xi=\xi_0\exp(G_\omega(\omega,\mu)t),\ \eta=\eta_0\exp(-G_\omega(\omega,\mu)t) \tag{6}$$

From (4) and (6) we find that the part of the complex separatrix S_μ lying in the region U_μ has the following parametric equation ($|\xi|<\varepsilon$ is the parameter):

$$x=\Phi(\xi,0,t,\mu)\ (t\in\Pi) \tag{7}$$

We shall assume, without loss of generality, that the value $x^*(\Gamma(0))$ is sufficiently close to x_0 and can be written in the form $x^*(\Gamma(0))=\Phi_0(\xi^*,0)$, $|\xi^*|<\varepsilon$. (This can be achieved by replacing the path Γ by $\Gamma_1^{-1}\Gamma\Gamma_1$ where $\Gamma_1:[0,1]\to R$ is a path such that $\Gamma_1(1)=\Gamma(0)$ and $\Gamma_1(0)$ is sufficiently small). Then, for sufficiently small $|\mu|$ the asymptotic solution $x=x_a(\xi_0,t,\mu)=\Phi(\xi_0\exp(G_\omega(0,\mu)t),0,t,\mu)$ can be analytically continued along the path $\Gamma+t_0(\xi_0)$, $t_0(\xi_0)=\lambda^{-1}\ln(\xi^*/\xi_0)-\Gamma(0)$.

Let $\bar{x}(\xi_0,\mu)=x_a(\xi_0,\Gamma(1)+t_0(\xi_0),\mu)$. We shall show that for sufficiently small $|\mu|\neq 0$ the curve (ξ_0 is a parameter)

$$x=\bar{x}(\xi_0,\mu),\ t=\bar{t}(\xi_0)=\Gamma(1)+t_0(\xi_0)\ \ (\bar{t}(\xi_0)\in\Pi) \tag{8}$$

intersects transversally the surface (7), and hence the complex separatrix S_μ self-intersects transversally. Let us write the equation of a part of the surface (7) in the form solved with respect to the coordinate

$$\psi=H_0(x),\ \psi=\chi(\xi,t,\mu)=H_0(\Phi(\xi,0,t,\mu))=H_0(x_0)+\mu\chi_1(\xi,t)+o(\mu)\ (t\in\Pi,\ |\xi-\xi^*|<\delta,\ \delta>0),$$

and the equation of the curve (8) in the parametric form

$$\xi=\bar{\xi}(\xi_0,\mu)=\xi^*+O(\mu),\ t=\bar{t}(\xi_0)$$

$$\psi=\bar{\psi}(\xi_0,\mu)=\chi(\xi^*+O(\mu),\bar{t}(\xi_0),\mu)+\int_{\Gamma+t_0(\xi_0)}\{H_0,H\}(x_a(t,\xi_0,\mu)t,\mu)\,dt=H_0(x_0)+\mu\chi^1(\xi^*,\bar{t}(\xi_0))+\mu h^1(t_0(\xi_0))+o(\mu)\quad(\bar{t}(\xi_0)\in\Pi)$$

To prove that a transversal intersection exists, it is sufficient to confirm that the equation

$$\bar{\psi}(\xi_0,\mu)-\chi(\bar{\xi}(\xi_0,\mu),\bar{t}(\xi_0),\mu)=\mu h^1(t_0(\xi_0))+o(\mu)=0\quad(t_0(\xi_0)\in\Pi)$$

has, for sufficiently small $|\mu|\neq 0$, a simple root, and this follows from the fact that the function h^1 has a simple zero in Π.

The nonexistence of an analytic first integral of the system (1) follows from the fact that within the region U_μ the integral must be an analytic function of the variable $\omega=\xi\eta$ /2/, and at the same time it must have a constant value on the curve (8) intersecting transversally the surface $\omega=0$, Q.E.D.

Example. We consider a problem of motion of a mathematical pendulum the length of which depends periodically on time. Here $M^2=C^1\times S_c^1$ is a direct product of the complex straight line C^1 with coordinate x_1 and the complex circumference S_c^1 with coordinate $x_2\ (\mathrm{mod}\ 2\pi)$; $\omega^2=dx_1\wedge dx_2$. The Hamiltonian has the form

$$H(x,t)={}^1/_2x_1^2+a(t)\cos x_2,\ a(t+2\pi)=a(t)$$

Let $a(t)=1+\mu\cos t$, μ be a small parameter. Then

$$H_0(x) = {}^1/_2 x_1^2 + \cos x_2, \quad H_1(x, t) = \cos t \cos x_2$$

The system with the Hamiltonian H_0 has a fixed hyperbolic point $x_0(0, 0)$. The asymptotic solution emerging from this point is single-valued, meromorphic and has the poles $a_k = i(\pi/2 + k\pi)$ (k is an integer).

$$x = x^*(t) : x_1 = \frac{2}{\operatorname{ch} t}, \quad \sin x_2 = -2\frac{\operatorname{sh} t}{\operatorname{ch}^2 t}, \quad \cos x_2 = 1 - \frac{2}{\operatorname{ch}^2 t}$$

Let $\Gamma : [0, 1] \to C, \Gamma(0) = \Gamma(1) \in R$ be a closed path going round the pole $a_0 = \pi i/2$ in the positive direction. The function

$$h^1(t_0) = \int_\Gamma \frac{\partial H_1}{\partial t}(x^*(t), t + t_0)\, dt = -\int_\Gamma \sin(t + t_0)\left(1 - \frac{2}{\operatorname{ch}^2 t}\right) dt = -4\pi i \cos(\pi i/2 + t_0)$$

has simple zeros, therefore by virtue of the theorem proved above, the complex separatrix S_μ self-intersects transversally at sufficiently small $|\mu| \neq 0$ and the system with the Hamiltonian H has no analytic first integral in $N^3 = M^2 \times S_c^1$.

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